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ASYMPTOTIC SOLUTION OF ONE CLASS OF SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEMS*

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Under certain conditions there is constructed and justified an asymptotic expansion in powers of a small parameter, of the solution of the problem of minimizing a quadratic functional on the trajectories of a singularly perturbed linear system not solved relative to the derivative, with fixed endpoints and fixed time. The limit passage, as the small parameter tends to zero, of the solution of the perturbed problem to the solution of a degenerate problem is established.

1. We examine the following classical fixed-time optimal control problem; find a continuous r-dimensional function $u(t)$ minimizing the functional

$$I(u)=\frac{1}{2}\int_0^T(\langle x(t),\ Qx(t)\rangle+\langle u(t),\ Ru(t)\rangle)\,dt \tag{1.1}$$

on the trajectories of the equation

$$\begin{gathered}(A+\varepsilon B)\,x^{\cdot}(t)=Cx(t)+Du(t),\ \ x(0)=x^{\circ},\ \ x(T)=x^{T}\\ x(t)\in R^{m},\ \ A,B,C:R^{m}\to R^{m},\ D:R^{r}\to R^{m}\end{gathered} \tag{1.2}$$

Here $\varepsilon>0$ is a small parameter, $T>0$ is a fixed number, all matrices are constant, R is a positive-definite symmetric $r\times r$-matrix, Q is a positive-semidefinite symmetric $m\times m$-matrix, matrix A is singular, and the matrix $A+\varepsilon B$ is invertible for sufficiently small $\varepsilon\neq 0$; $\langle\cdot,\cdot\rangle$ denotes the scalar product. For

$$A=\begin{Vmatrix}0&0\\0&E\end{Vmatrix},\quad B=\begin{Vmatrix}E&0\\0&0\end{Vmatrix} \tag{1.3}$$

(E is the unit matrix) problem (1.1), (1.2) was analyzed in /1/, where the zero approximation of the solution was constructed. Equations not solved relative to the derivative are encountered, for example, in economics (the input-output equation /2/). Using Pontriagin's maximum principle /3/, we arrive at the two-point boundary-value problem

$$\begin{gathered}(A+\varepsilon B)\,x^{\cdot}(t,\varepsilon)=Cx(t,\varepsilon)+S\psi(t,\varepsilon);\ \ x(0,\varepsilon)=x^{\circ}\\ x(T,\varepsilon)=x^{T}\\ (A'+\varepsilon B')\,\psi^{\cdot}(t,\varepsilon)=Qx(t,\varepsilon)-C'\psi(t,\varepsilon)\\ S=DR^{-1}D',\ \psi(t,\varepsilon)=(A'+\varepsilon B')^{-1}\varphi(t)\end{gathered} \tag{1.4}$$

where $\varphi(t)$ is the adjoint variable and the prime denotes transposition. Here the optimal control takes the form

$$u(t,\varepsilon)=R^{-1}D'\psi(t,\varepsilon) \tag{1.5}$$

2. Let the kernel of matrix A be one-dimensional. Then we can take it that by nonsingular transformations of system (1.2) the matrix A can be brought to the form diag $\{J,E\}$, where J is the Jordan cell corresponding to the zero eigenvalue. By e_1 we denote the eigenvector of matrix A, corresponding to the zero eigenvalue, and by $e_2,\ldots,e_n$ the corresponding chain of associated eigenvalues. We assume as well that

$$C_{n1}=\langle Ce_1,e_n\rangle\neq 0,\ Q_{11}=\langle Qe_1,e_1\rangle\neq 0 \tag{2.1}$$

where C_{ij} denotes the element in the i-th row and j-th column of matrix C. Henceforth, for

*Prikl.Matem.Mekhan., Vol.47,No.3,pp.363-371,1983 the i-th components of the vectors x and Cx we shall use the notation x^i and $-(Cx)^i$, respectively: $i = 1, 2, \ldots$; $[x]^i$ is a vector from R^{m-1}, obtained from a vector $x \in R^m$ by deletion of the i-th component; $[C]^{ij}$ is an $(m-1) \times (m-1)$-matrix obtained from the $m \times m$-matrix C by deletion of the i-th row and j-th column.

Let us consider Eq.(1.2) with $\varepsilon = 0$, i.e., the unperturbed equation relative to $x^{\cdot}(t)$. On the strength of condition (2.1), we express x^1 from the algebraic equation and we substitute the resultant expression in the differential equation relative to $[x]^1$. We obtain

$$[x^{\cdot}]^1 = C_0[x]^1 + D_0 u; \quad C_0[x]^1 = [C]^{n1}[x]^1 - [Ce_1]^n \langle [C'e_n]^1, \ [x]^1 \rangle / C_{n1}, \quad D_0 u = [Du]^n - [Ce_1]^n (Du)^n / C_{n1} \tag{2.2}$$

We assume that the degenerate system (2.2) is completely controllable, i.e.,

$$\operatorname{rank}\,(D_0, C_0 D_0, \ldots, C_0^{m-2} D_0) = m - 1 \tag{2.3}$$

and that the condition

$$D'e_n \neq 0 \tag{2.4}$$

is fulfilled. Under these conditions, by using the change of variables

$$\xi = [x]^1 + \varepsilon H \eta, \quad \eta = x^1 + G'[x]^1$$

to lead system (1.2) to the form

$$\xi^{\cdot} = (C_0 + O(\varepsilon))\xi + (D_0 + O(\varepsilon))u, \quad \eta^{\cdot} = ((C_{n1} + O(\varepsilon))\eta + (Du)^n + O(\varepsilon)u)/(\varepsilon^p \nu)$$

$$\nu = (-1)^{p-1} \langle B(A'B)^{p-1} e_1, e_n \rangle, \quad p = \min_{\langle B(A'B)^{i-1} e_1, e_n \rangle \neq 0} i$$

we can show that system (1.2) is completely controllable for sufficiently small $\varepsilon \neq 0$. The existence of the number $p \geqslant 1$ follows (*) from the condition of invertibility of the matrix $A + \varepsilon B$ for sufficiently small ε. For matrices A and B of form (1.3) the controllability of the perturbed system has been proved in /4/. Using the result in /5/ we can prove that under the assumptions made on the properties of matrices Q and R, a boundary-value problem of type (1.4) has a unique solution if and only if system (1.2) is controllable. For sufficiently small $\varepsilon \neq 0$ the complete controllability of system (1.2) follows from conditions (2.1), (2.3), (2.4) and, therefore, the optimal control (1.5) has been defined uniquely.

3. System (1.4) is singularly perturbed /6/. We shall seek the asymptotics of the solution of problem (1.4) for an arbitrary integer $q \geqslant 0$ in the form

$$x(t, \varepsilon) = \sum_{j=0}^{q} \varepsilon^j (x_j(t) + \Pi_j x(\tau_0) + Q_j x(\tau_1)) + r_q x(t, \varepsilon) \tag{3.1}$$

$$\psi(t, \varepsilon) = \sum_{j=0}^{q} \varepsilon^j (\psi_j(t) + \Pi_j \psi(\tau_0) + Q_j \psi(\tau_1)) + r_q \psi(t, \varepsilon);$$

$$\tau_0 = \frac{t}{\varepsilon^p}, \quad \tau_1 = \frac{t - T}{\varepsilon^p}$$

where all functions are continuously differentiable, $\Pi_j x(\tau_0), \Pi_j \psi(\tau_0)$ are boundary-layer functions in a neighborhood of $t = 0$, $Q_j x(\tau_1), Q_j \psi(\tau_1)$ are boundary-layer functions in a neighborhood of $t = T$, while the remaining terms $r_q x(t, \varepsilon), r_q \psi(t, \varepsilon)$ have the estimates

$$|r_q x(t, \varepsilon)|, \ |r_q \psi(t, \varepsilon)| \leqslant c\varepsilon^{q+1} \tag{3.2}$$

$$|r_q^{\cdot} x(t, \varepsilon)|, \ |r_q^{\cdot} \psi(t, \varepsilon)| \leqslant c\varepsilon^{q+1-p}$$

Here and further c denotes a positive constant not dependent on ε, t, τ, s; $|\cdot|$ denotes a norm. The next lemma is easily proved.

Lemma 1. In order that the functions $x(t, \varepsilon), \psi(t, \varepsilon)$, having expansions (3.1) with estimates (3.2) for any integer $q \geqslant 0$, be a solution of system (1.4), it is necessary and sufficient that for $j = 0, 1, \ldots, q$ the functions $x_j(t), \psi_j(t)$ be solutions of the system of equations

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*) For example, see, ZUBOVA S.P., Singular perturbation of linear differential equations not solved relative to the derivative. Dissertation for the scientific degree of Candidate of Physico-Mathematical Sciences, Voronezh. State Univ., 1973.

$$A x_j^{\cdot}(t) = C x_j(t) + S\psi_j(t) - B\dot{x}_{j-1}(t) \qquad (3.3)$$
$$A'\psi_j^{\cdot}(t) = Q x_j(t) - C'\psi_j(t) - B'\dot{\psi}_{j-1}(t)$$

the functions $\Pi_j x(\tau_0)$, $\Pi_j\psi(\tau_0)$ be solutions of the system of equations

$$A\frac{d\Pi_j x(\tau_0)}{d\tau_0} = -B\frac{d\Pi_{j-1}x(\tau_0)}{d\tau_0} + C\Pi_{j-p}x(\tau_0) + S\Pi_{j-p}\psi(\tau_0) \qquad (3.4)$$
$$A'\frac{d\Pi_j\psi(\tau_0)}{d\tau_0} = -B'\frac{d\Pi_{j-1}\psi(\tau_0)}{d\tau_0} - C'\Pi_{j-p}\psi(\tau_0) + Q\Pi_{j-p}x(\tau_0)$$

the functions $Q_j x(\tau_1)$, $Q_j\psi(\tau_1)$ satisfy system (3.4) with the substitution of τ_0 by τ_1, $\Pi_j x(\tau_0)$ by $Q_j x(\tau_1)$, $\Pi_j\psi(\tau_0)$ by $Q_j\psi(\tau_1)$, and the remaining terms $r_q x(t,\varepsilon)$, $r_q\psi(t,\varepsilon)$ satisfy the system

$$(A+\varepsilon B) r_q^{\cdot} x(t,\varepsilon) = C r_q x(t,\varepsilon) + S r_q\psi(t,\varepsilon) + F_1(t,\varepsilon) \qquad (3.5)$$
$$(A'+\varepsilon B') r_q^{\cdot}\psi(t,\varepsilon) = Q r_q x(t,\varepsilon) - C' r_q\psi(t,\varepsilon) + F_2(t,\varepsilon)$$
$$F_1(t,\varepsilon) = -\varepsilon^{q+1} B x_q^{\cdot}(t) + C\sum_{j=q+1-p}^{q}\varepsilon^j(\Pi_j x(\tau_0) + Q_j x(\tau_1)) +$$
$$S\sum_{j=q+1-p}^{q}\varepsilon^j(\Pi_j\psi(\tau_0) + Q_j\psi(\tau_1)) - \varepsilon^{q+1-p}B\left(\frac{d\Pi_q x(\tau_0)}{d\tau_0} + \frac{dQ_q x(\tau_1)}{d\tau_1}\right)$$
$$F_2(t,\varepsilon) = -\varepsilon^{q+1} B'\psi_q^{\cdot}(t) - C'\sum_{j=q+1-p}^{q}\varepsilon^j(\Pi_j\psi(\tau_0) + Q_j\psi(\tau_1)) +$$
$$Q\sum_{j=q+1-p}^{q}\varepsilon^j(\Pi_j x(\tau_0) + Q_j x(\tau_1)) - \varepsilon^{q+1-p}B'\left(\frac{d\Pi_q\psi(\tau_0)}{d\tau_0} + \frac{dQ_q\psi(\tau_1)}{d\tau_1}\right)$$

In all expressions the functions with indices $j<0$ are taken equal to zero.

The function $x(t,\varepsilon)$ must satisfy the boundary conditions from (1.4). Therefore, the boundary conditions for the functions occurring in expansion (3.1) are naturally specified in the following manner:

$$x_0(0) + \Pi_0 x(0) = x^\circ,\ \ x_j(0) + \Pi_j x(0) = 0,\ \ j\neq 0 \qquad (3.6)$$
$$x_0(T) + Q_0 x(0) = x^T,\ \ x_j(T) + Q_j x(0) = 0,\ \ j\neq 0$$
$$r_q x(0,\varepsilon) = -\sum_{j=0}^{q}\varepsilon^j Q_j x\left(-\frac{T}{\varepsilon^p}\right);\ r_q x(T,\varepsilon) = -\sum_{j=0}^{q}\varepsilon^j\Pi_j x\left(\frac{T}{\varepsilon^p}\right) \qquad (3.7)$$

Let us find the first term of asymptotics (3.1). Analogously to /7/ we can show that systems (3.4) are solvable for $j = 0, 1, \ldots, p-1$ and their solutions have the form

$$\Pi_j x(\tau_0) = \sum_{k=0}^{j}(-1)^{j-k} b_k^1(\tau_0)(A'B)^{j-k}e_1$$
$$\Pi_j\psi(\tau_0) = \sum_{k=0}^{j}(-1)^{j-k} b_k^2(\tau_0)(AB')^{j-k}e_n$$

where $b_k^1(\tau_0)$, $b_k^2(\tau_0)$ are as yet unknown continuously differentiable functions. The equalities

$$\left\langle -B\frac{d\Pi_{p-1}x(\tau_0)}{d\tau_0} + C\Pi_0 x(\tau_0) + S\Pi_0\psi(\tau_0),\ \ e_n\right\rangle \equiv 0$$
$$\left\langle -B'\frac{d\Pi_{p-1}\psi(\tau_0)}{d\tau_0} - C'\Pi_0\psi(\tau_0) + Q\Pi_0 x(\tau_0),\ \ e_1\right\rangle \equiv 0$$

are the solvability conditions for system (3.4) with $j = p$. From the latter relations, with due regard to the preceding expressions, and from the definition of number p we obtain the system

$$\frac{db_0^1(\tau_0)}{d\tau_0} = \frac{C_{n1}b_0^1(\tau_0)}{\nu} + \frac{S_{nn}b_0^2(\tau_0)}{\nu} \qquad (3.8)$$
$$\frac{db_0^2(\tau_0)}{d\tau_0} = \frac{Q_{11}b_0^1(\tau_0)}{\nu} - \frac{C_{n1}b_0^2(\tau_0)}{\nu}$$

In the same way we find that $Q_0x(\tau_1) = a_0^1(\tau_1)e_1$, $Q_0\psi(\tau_1) = a_0^2(\tau_1)e_n$, where $a_0^1(\tau_1)$, $a_0^2(\tau_1)$ satisfy exactly the same system as for the functions $b_0^1(\tau_0)$, $b_0^2(\tau_0)$. From (3.6) we have

$$x_0(0) = x^\circ - b_0^1(0)e_1, \quad x_0(T) = x^T - a_0^1(0)e_1 \tag{3.9}$$

Hence we define $[x_0(0)]^1$, $[x_0(T)]^1$.

Lemma 2. Under the assumptions made on the properties of the matrices the system

$$Ay^\cdot = Cy + Sz, \quad A'z^\cdot = Qy - C'z \tag{3.10}$$

with the zero boundary conditions $[y(0)]^1 = [y(T)]^1 = 0$ is uniquely solvable.

Proof. We multiply the first equation of system (3.10) scalarly by z and the second by y and we add the results. Integrating the resulting equality with respect to t from 0 to T and accounting for the zero boundary conditions, we find

$$\int_0^T (\langle Sz, z\rangle + \langle Qy, y\rangle)\, dt = 0$$

Because of the positive semidefiniteness of matrices S, Q from the last equality follows $Sz = 0$, $Qy = 0$. Therefore, from system (3.10) we have

$$Ay^\cdot = Cy, \quad A'z^\cdot = -C'z$$

From the first equation of this system, by virtue of the zero boundary condition for y, we obtain $y(t) \equiv 0$. From the second equation we obtain

$$z^n = -\langle [z]^n, [Ce_1]^n\rangle / C_{n1}$$

Taking this relation into account, we have

$$\langle Sz, z\rangle = \langle D_0R^{-1}D_0'[z]^n, [z]^n\rangle$$

By $\Phi(t, s)$ we denote the fundamental matrix for matrix C_0. Then

$$[z(t)]^n = \Phi'(0, t)\,[z(0)]^n$$

$$\int_0^T \langle Sz, z\rangle\, dt = \int_0^T \langle D_0R^{-1}D_0'[z(t)]^n, \ [z(t)]^n\rangle\, dt =$$

$$\int_0^T \langle \Phi(0, t)D_0R^{-1}D_0'\Phi'(0, t)\,[z(0)]^n, \ [z(0)]^n\rangle\, dt = 0$$

From the latter equality, because of the complete controllability of the degenerate system (2.2) and the positive definiteness of matrix R^{-1} we have $[z(0)]^n = 0$. Hence $[z(t)]^n \equiv 0$, $z^n(t) \equiv 0$, i.e. $z(t) \equiv 0$.

Lemma 3. If the homogeneous problem

$$y^\cdot = Cy, \quad My(0) - Ny(T) = 0$$

has no nontrivial solution, then the inhomogeneous problem

$$y^\cdot = Cy + g(t), \quad My(0) - Ny(T) = d$$

where $g(t)$ is any prescribed continuous function and d is a prescribed constant vector, has a unique solution.

Proof. The general solution of the inhomogeneous equation is given by the formula

$$y(t) = \exp(Ct)\,y_0 + \int_0^t \exp(C(t-s))\,g(s)\,ds$$

and satisfies the prescribed condition if and only if

$$My_0 - N(\exp(CT)\,y_0 + \int_0^T \exp(C(T-s))\,g(s)\,ds) = d$$

Since the homogeneous problem does not have a nontrivial solution, from /8/ it follows that the matrix $M - N\exp(CT)$ is not singular. Therefore, y_0 is uniquely determined from the preceding equation. Hence, the solution of the inhomogeneous problem has been uniquely determined. The assertion of the last lemma was proved in /8/ for $d = 0$.

We can now prove the unique solvability of the system

$$Ay^\cdot = Cy + Sz + g_1, \quad A'z^\cdot = Qy - C'z + g_2$$

with prescribed boundary conditions $[y(0)]^1$, $[y(T)]^1$. If in this system we pass to the coordinate notation, from the finite relations we can express y^1, z^n in terms of $[y]^1, [z]^n, g_1, g_2$.

Substituting the resultant expressions into the system of differential equations relative to functions $[y]^1$, $[z]^n$, we arrive at a system of inhomogeneous differential equations for the functions $[y]^1$, $[z]^n$ with the prescribed boundary conditions $[y(0)]^1$, $[y(T)]^1$. The unique solvability of the homogeneous boundary-value problem for $[y]^1, [z]^n$ follows from Lemma 2. By virtue of Lemma 3 the inhomogeneous boundary-value problem for $[y]^1, [z]^n$ is uniquely solvable. Therefore, the original boundary-value problem is uniquely solvable.

On the basis of the preceding arguments, $x_0(t)$, $\psi_0(t)$ are uniquely determined from system (3.3). From (3.9) we now can find $b_0^1(0)$, $a_0^1(0)$. One condition for system (3.8) is well known, while a second is the condition that $b_0^1(\tau_0)$ tends to zero as $\tau_0 \to +\infty$. Thus, $b_0^1(\tau_0)$, $b_0^2(\tau_0)$ have been determined, and, hence, also $\Pi_0 x(\tau_0)$, $\Pi_0\psi(\tau_0)$. The functions $Q_0 x(\tau_1)$, $Q_0\psi(\tau_1)$ are found analogously. Thus, the first term of the asymptotics (3.1) has been constructed.

Lemma 4. Let $q \geqslant p$. For any $j, j = 0, 1, \ldots, q-p$, from the equations in Lemma 1 we can uniquely determine the functions $x_j(t)$, $\psi_j(t)$, $\Pi_j x(\tau_0)$, $\Pi_j\psi(\tau_0)$, $Q_j x(\tau_1)$, $Q_j\psi(\tau_1)$; the equations for the functions $\Pi_{j+p}x(\tau_0)$, $\Pi_{j+p}\psi(\tau_0)$, $Q_{j+p}x(\tau_1)$, $Q_{j+p}\psi(\tau_1)$ are solvable, the components of $\Pi_j x(\tau_0)$, $\Pi_j\psi(\tau_0)$ have the form $e^{-\lambda\tau_0}p(\tau_0)$, the components of the functions $Q_j x(\tau_1)$, $Q_j\psi(\tau_1)$ have the form $e^{\lambda\tau_1}q(\tau_1)$, where $p(\tau_0)$, $q(\tau_1)$ are certain polynomials of both arguments, $\lambda = ((C_{n1})^2 + Q_{11}S_{nn})^{1/2} |\nu|^{-1}$.

The differential equations for determining the functions $b_k^i(\tau_0)$, $a_k^i(\tau_1)$ $(i = 1, 2)$ are found from the solvability conditions for the equations for $\Pi_{k+p}x(\tau_0)$, $\Pi_{k+p}\psi(\tau_0)$, $Q_{k+p}x(\tau_1)$, $Q_{k+p}\psi(\tau_1)$. Therefore, in order to find all the boundary-layer functions in expansions (3.1) it is necessary to write these expansions to within ε^{q+p}.

4. Now we can prove the estimates (3.2) for the solution of system (3.5) with conditions (3.7). To do this we split the system (3.5) into two systems:

$$\begin{aligned}(E + \varepsilon[B]^{n1})[z^{\cdot}]^1 &= [C]^{n1}[z]^1 + [S]^{nn}[y]^n + \\ &[F_1 - \varepsilon Be_1(z^1)^{\cdot} + Ce_1z^1 + Se_ny^n]^n \\ (E + \varepsilon[B']^{1n})[y^{\cdot}]^n &= [Q]^{11}[z]^1 - [C']^{1n}[y]^n + \\ &[F_2 - \varepsilon B'e_n(y^n)^{\cdot} + Qe_1z^1 - C'e_ny^n]^1\end{aligned} \tag{4.1}$$

$$\varepsilon(Bz^{\cdot})^n = (Cz)^n + (Sy)^n + F_1^n, \quad \varepsilon(B'y^{\cdot})^1 = (Qz)^1 - (C'y)^1 + F_2^1 \tag{4.2}$$

Here

$$z(t, \varepsilon) = r_q x(t, \varepsilon), \quad y(t, \varepsilon) = r_q\psi(t, \varepsilon)$$

From (4.2) we express z^1, y^n and we substitute the expressions obtained into (4.1). We arrive at a system of form

$$\left\|\begin{matrix}[z^{\cdot}]^1 \\ [y^{\cdot}]^n\end{matrix}\right\| = F(\varepsilon)\left\|\begin{matrix}[z]^1 \\ [y]^n\end{matrix}\right\| + \varepsilon\frac{d}{dt}\left\|\begin{matrix}g_1(z, y, \varepsilon, t) \\ g_2(z, y, \varepsilon, t)\end{matrix}\right\| + \left\|\begin{matrix}f_1(z^1, y^n, \varepsilon, t) \\ f_2(z^1, y^n, \varepsilon, t)\end{matrix}\right\| \tag{4.3}$$

We do not write out the exact expressions for the coefficients of the last system because of their cumbersomeness. When estimating the remainder term we denote by $f_i(z^1, y^n, \varepsilon, t)$ a vector-valued function of dimension 1 or $m-1$, for which the inequalities

$$|f_i(z^1, y^n, \varepsilon, t)| \leqslant c\varepsilon(|z^1(t, \varepsilon)|_{C[0,T]} + |y^n(t, \varepsilon)|_{C[0,T]}) + c\varepsilon^{q+1-p} \quad (i = 1, 2)$$

are valid. (From the form of functions $F_1(t, \varepsilon)$, $F_2(t, \varepsilon)$ and the properties of the functions occurring in expansions (3.1) it follows that the inequalities $|F_i(t, \varepsilon)| \leqslant c\varepsilon^{q+1-p}$ $(i = 1, 2)$ hold). By virtue of Lemma 2, the boundary-value problem (3.5), (3.7) is uniquely solvable when $\varepsilon = 0$. Therefore, system (4.3) with prescribed boundary conditions $[z(0, \varepsilon)]^1$, $[z(T, \varepsilon)]^1$ is uniquely solvable for sufficiently small ε and, using the Green's function $G(t, s, \varepsilon)$, its solution can be written as

$$\begin{aligned}\left\|\begin{matrix}[z(t, \varepsilon)]^1 \\ [y(t, \varepsilon)]^n\end{matrix}\right\| &= \int_0^T G(t, s, \varepsilon)\left(\varepsilon\frac{d}{ds}\left\|\begin{matrix}g_1(z, y, \varepsilon, s) \\ g_2(z, y, \varepsilon, s)\end{matrix}\right\| + \left\|\begin{matrix}f_1(z^1, y^n, \varepsilon, s) \\ f_2(z^1, y^n, \varepsilon, s)\end{matrix}\right\|\right)ds + \\ &\exp(F(\varepsilon)t)V^{-1}\left\|\begin{matrix}[z(0, \varepsilon)]^1 \\ [z(T, \varepsilon)]^1\end{matrix}\right\|, \quad V = \left\|\begin{matrix}E & 0 \\ 0 & 0\end{matrix}\right\| - \left\|\begin{matrix}0 & 0 \\ -E & 0\end{matrix}\right\|\exp(F(\varepsilon)T)\end{aligned} \tag{4.4}$$

It can be proved that the estimates

$$|G(t, s, \varepsilon)|, \quad \left|\frac{dG(t, s, \varepsilon)}{ds}\right| \leqslant c$$

are valid. In expression (4.4) we apply the integration by parts formula to the terms containing the derivatives (allowing for the discontinuities of the function $G(t, s, \varepsilon)$ when $t = s$). Further, taking into account the estimates for functions $Q_j x(-T/\varepsilon^p)$, $\Pi_j x(T/\varepsilon^p)$ and the estimates for the function $G(t, s, \varepsilon)$, we get that

$$[z(t,\varepsilon)]^1 = f_1(z^1, y^n, \varepsilon, t), \quad [y(t,\varepsilon)]^n = f_2(z^1, y^n, \varepsilon, t) \tag{4.5}$$

We pass on to Eqs.(4.2). Using the preceding relations and the definition of number p, we can reduce them to the form

$$(z^1)^\cdot = \frac{C_{n1}}{\nu\varepsilon^p} z^1 + \frac{S_{nn}}{\nu\varepsilon^p} y^n + \frac{1}{\varepsilon^p} f_1(z^1, y^n, \varepsilon, t) \tag{4.6}$$
$$(y^n)^\cdot = \frac{Q_{11}}{\nu\varepsilon^p} z^1 - \frac{C_{n1}}{\nu\varepsilon^p} y^n + \frac{1}{\varepsilon^p} f_2(z^1, y^n, \varepsilon, t)$$

If in this system we treat f_1, f_2 as the inhomogeneity, then the homogeneous system with prescribed boundary conditions $z^1(0,\varepsilon)$, $z^1(T,\varepsilon)$ is uniquely solvable and we can write the solution of system (4.6) with prescribed boundary conditions $z^1(0,\varepsilon)$, $z^1(T,\varepsilon)$, using the Green's function $G(t,s,\varepsilon^p)$, in the form

$$\left\|\begin{matrix} z^1(t,\varepsilon) \\ y^n(t,\varepsilon)\end{matrix}\right\| = \frac{1}{\varepsilon^p}\int_0^T G(t,s,\varepsilon^p)\left\|\begin{matrix} f_1(z^1, y^n, \varepsilon, s) \\ f_2(z^1, y^n, \varepsilon, s)\end{matrix}\right\| ds + \exp(Kt)W^{-1}\left\|\begin{matrix} z^1(0,\varepsilon) \\ z^1(T,\varepsilon)\end{matrix}\right\|$$

$$K = \left\|\begin{matrix} \dfrac{C_{n1}}{\nu\varepsilon^p} & \dfrac{S_{nn}}{\nu\varepsilon^p} \\ \dfrac{Q_{11}}{\nu\varepsilon^p} & -\dfrac{C_{n1}}{\nu\varepsilon^p}\end{matrix}\right\|, \quad W = \left\|\begin{matrix} 1 & 0 \\ 0 & 0\end{matrix}\right\| - \left\|\begin{matrix} 0 & 0 \\ -1 & 0\end{matrix}\right\| \exp(KT)$$

From the form of the function $G(t,s,\varepsilon^p)$ follows the validity of the inequality

$$\frac{1}{\varepsilon^p}\int_0^T |G(t,s,\varepsilon^p)|\, ds \leqslant c$$

By virtue of this inequality and the form of the matrix $\exp(Kt)W^{-1}$ we have

$$z^1(t,\varepsilon) = f_1(z^1, y^n, \varepsilon, t), \quad y^n(t,\varepsilon) = f_2(z^1, y^n, \varepsilon, t)$$

whence from relations (4.5) it follows that

$$|r_q x(t,\varepsilon)|_{C[0,T]}, \quad |r_q\psi(t,\varepsilon)|_{C[0,T]} \leqslant c\varepsilon^{q+1-p}$$

From the differential equations for $r_q x(t,\varepsilon)$, $r_q\psi(t,\varepsilon)$ we obtain the estimates

$$|r_q^\cdot x(t,\varepsilon)|_{C[0,T]}, \quad |r_q^\cdot \psi(t,\varepsilon)|_{C[0,T]} \leqslant c\varepsilon^{q+1-2p}$$

In order to obtain estimates (3.2) we need to write the expansions for x, ψ to within ε^{q+p} and to take advantage of the estimates proved above for $r_{q+p}x(t,\varepsilon)$, $r_{q+p}\psi(t,\varepsilon)$ (in the preceding inequalities write $q+p$ instead of q).

5. Using the asymptotics for the function $\psi(t,\varepsilon)$ we obtain the asymptotic expansion of the optimal control (1.5) in a power series in ε. Since the asymptotics for the optimal control and the optimal trajectory have been constructed, we can write an asymptotic expansion in powers of ε for the minimal value of the functional

$$I(u) = \sum_{j=0}^{q} \varepsilon^j I_j + O(\varepsilon^{q+1})$$

We investigate the solution's behavior as $\varepsilon \to 0$. From the form of the asymptotics it follows that $x(t,\varepsilon) \to x_0(t), \psi(t,\varepsilon) \to \psi_0(t)$ in the metrics of $C[T_1, T_2]$ $(0 < T_1 < T_2 < T)$ and of $L_1[0,T]$. In addition, since $[\Pi_0 x(\tau_0)]^1 = [Q_0 x(\tau_1)]^1 \equiv 0$, $[\Pi_0\psi(\tau_0)]^n = [Q_0\psi(\tau_1)]^n \equiv 0$, we have $[x(t,\varepsilon)]^1 \to [x_0(t)]^1$, $[\psi(t,\varepsilon)]^n \to [\psi_0(t)]^n$ in the metric of $C[0,T]$.

We pose a degenerate control problem. Find a control $u(t)$ minimizing the functional

$$\bar{I}(u) = \frac{1}{2}\int_0^T (\langle \bar{x}(t), Q\bar{x}(t)\rangle + \langle u(t), \quad Ru(t)\rangle)\, dt \tag{5.1}$$

on the trajectories of the equation

$$A\bar{x}^\cdot(t) = C\bar{x}(t) + Du(t), \quad [\bar{x}(0)]^1 = [x^\circ]^1, \quad [\bar{x}(T)]^1 = [x^T]^1 \tag{5.2}$$

Using the method of proof of the sufficiency of the optimality conditions in /3/, we can prove that the optimal control $\bar{u}(t)$ for the degenerate problem is determined from the maximum principle

$$-{}^1/_2\langle\bar{u}(t),\ R\bar{u}(t)\rangle + \langle\psi_0(t),\ D\bar{u}(t)\rangle = \max_u(-{}^1/_2\langle u,\ Ru\rangle + \langle\psi_0(t),\ Du\rangle)$$

Since in the problem being analyzed no constraints are imposed on the control, from the preceding relation we find $\bar{u}(t) = R^{-1}D'\psi_0(t)$. It happens here that $\bar{x}(t) = x_0(t)$. From the form of the optimal control for the degenerate problem we obtain an assertion on the tending of the perturbed problem's solution to the solution of the degenerate one, in the metrics of $C[T_1, T_2]$ $(0 < T_1 < T_2 < T)$ and of $L_1[0, T]$. Moreover, the minimal value of the perturbed problem's functional $I(u)$ tends to the minimal value of the degenerate problem's functional $\bar{I}(u) = I_0$ as $\varepsilon \to 0$. Thus, we have proved the following theorem.

Theorem. Under the fulfillment of the above-listed conditions imposed on the matrices A, B, C, D, Q, R, for the problem (1.1), (1.2) there exist asymptotic expansions for the optimal trajectory, the optimal control and the minimal value of functional (1.1) in powers of ε. As $\varepsilon \to 0$ the solution of the perturbed problem (1.1), (1.2) tends, in the appropriate norms, to the solution of the degenerate problem (5.1), (5.2).

6. We apply the proposed algorithm for constructing the asymptotics to the example from /1/ with $\varepsilon = 0.1$

$$I(u) = \frac{1}{2}\int_0^1 (2(x^1)^2 + 4(x^2)^2 + (u)^2)\,dt$$

$$\varepsilon(x^1)^\cdot = 0.5x^1 - 1.5x^2 - u,\quad x^1(0) = 3,\quad x^1(1) = -1.3$$

$$(x^2)^\cdot = 1.5x^1,\ x^2(0) = 4,\ x^2(1) = 0.5$$

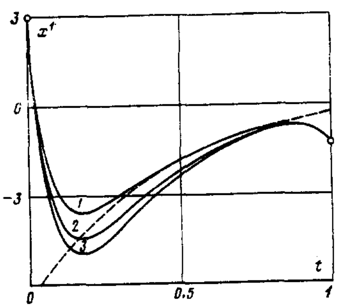

Fig.1

The calculation results are shown in the Fig.1. The dashed curve corresponds to the degenerate solution $\bar{x}^1(t)$; curve *1* corresponds to the solution in the zero approximation (to within $O(1)$) with due regard to the boundary layers; curve *2* corresponds to the solution in the first approximation (to within $O(\varepsilon)$); curve *3* corresponds to the exact solution $x^1(t,\varepsilon)$. It turned out that the solution in the second approximation (to within $O(\varepsilon^2)$) practically coincides with the exact solution.

We remark that in /1/ only the zero approximation was constructed, with the use of solutions of the Riccati equation.

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